Week of September 22nd

Real and Complex Numbers
Reciprocal spaces, X-ray diffraction

Exercise 1 :

1a. What are the real and imaginary parts of: *(i)* $\sqrt{-16}$; *(ii)* $\frac{1+3i}{2+i}$

1b. Find the modulus and argument of the following complex numbers, and express in the polar form: *(i)* $\frac{1}{\sqrt{2}}(1+i)$; *(ii)* $\frac{1+i\sqrt{3}}{1+i}$

1c. Simplify the following complex expressions: $\left(\frac{1+i}{1-i}\right)^{2025}$

1d.

(i) Using Euler's relation, show that:

$$\forall (x,y) \in \mathbb{R}^2, sin(x+iy) = sin(x)\,ch(y) + icos(x)sh\,(y)$$

(Reminder: $\forall y \in \mathbb{R}, ch(y) = \frac{e^y+e^{-y}}{2}$ and $sh(y) = \frac{e^y-e^{-y}}{2}$)

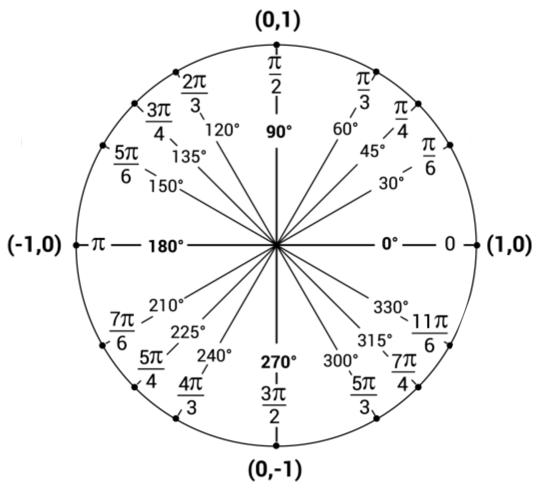
(ii) Use this expression to find one solution of the equation in $\mathbb{C}$: $sin(z) = 2$.

Exercise 2 : Trigonometry and unit circle

2a. Place with a cross the following complex numbers on the unit circle below:

(i) $e^{i\frac{7\pi}{4}}$; (ii) $\frac{1}{\sqrt{2}}(1-i)$; (iii) $z = i \times \frac{\sqrt{3}+i}{2}$;

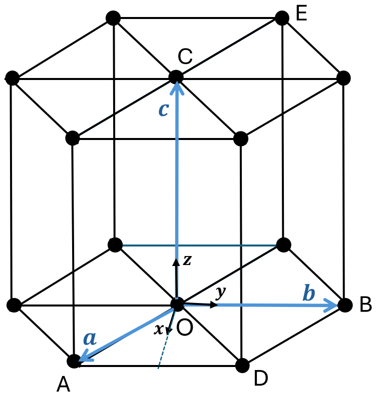
2b. Calculate the roots of the equation: $z^3 = \cos\left(\frac{3\pi}{2}\right) + isin\left(\frac{3\pi}{2}\right)$. Place them on the unit circle and show that they form an equilateral triangle.



Exercise 3: Distance between crystal planes in the hexagonal structure

We consider the hexagonal structure shown to the right. We represented the origin, the orthonormal basis $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$, and the Bravais lattice $\mathcal{B}_{(O,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}$, with:

- $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = a$, $\|\boldsymbol{c}\| = c$, and $a \neq c$ (where $\|\boldsymbol{a}\|$ is the norm of the vector $\boldsymbol{a}$);
- $\widehat{(\boldsymbol{a},\boldsymbol{b})} = \frac{2\pi}{3}$, $\widehat{(\boldsymbol{a},\boldsymbol{c})} = \widehat{(\boldsymbol{b},\boldsymbol{c})} = \frac{\pi}{2}$ (where $\widehat{(\boldsymbol{a},\boldsymbol{b})}$ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$).



3a.

(i) What are the coordinates of the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in the basis $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$?

(ii) What are the coordinates in the basis $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$ and $\mathcal{B}_{(O,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}$ of the points A,B, C, D and E shown on the schematic ?

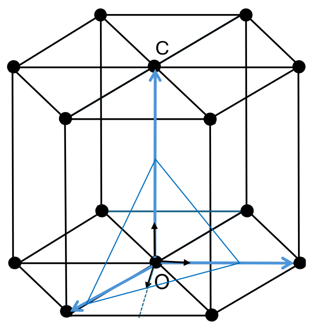
3b.

(i) What is the volume of the cell defined by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$?

(ii) Show that the reciprocal lattice vectors are given by, in the $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$ basis:

$$\boldsymbol{a}^* = \frac{4\pi}{a\sqrt{3}}\boldsymbol{x};\ \boldsymbol{b}^* = \frac{4\pi}{a\sqrt{3}}\left(\frac{1}{2}\boldsymbol{x} + \frac{\sqrt{3}}{2}\boldsymbol{y}\right);\ \boldsymbol{c}^* = \frac{2\pi}{c}\boldsymbol{z}$$

3c. We consider the crystal plane (hkl) (h,k,l three relative integers) shown on the schematic, that intercepts the $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ axis at positions $\frac{\boldsymbol{a}}{h}, \frac{\boldsymbol{b}}{k}, \frac{\boldsymbol{c}}{l}$ respectively.



(i) What is the normal to the plane in the orthonormal $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$ basis ?

(ii) Show that the equation of the plane in $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$ is given by:

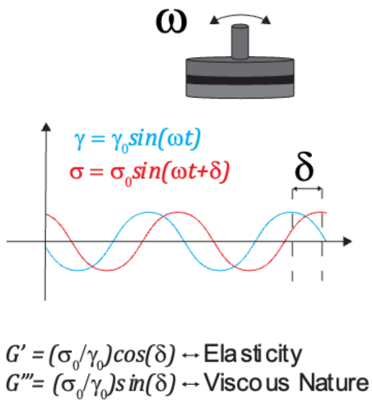
$$\mathcal{P} = \left\{ M\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \frac{(2h+k)}{a\sqrt{3}}x + \frac{k}{a}y + \frac{l}{c}z = 1 \right\}$$

3d. Assuming that the closest (hkl) plane parallel to $\mathcal{P}$ is the one passing through the origin, show that the distance between the (hkl) plane is given by:

$$d_{(hkl)} = \frac{1}{\sqrt{\frac{4}{3a^2}(h^2 + k^2 + hk) + \frac{l^2}{c^2}}}$$

Exercise 4 : Rheology and complex numbers

Dynamic or oscillatory rheology is a powerful technique to characterize the viscoelastic properties of materials (e.g. its viscosity η). A schematic is illustrated on the right picture: a cylindrical sample is placed between two parallel plates and a sinusoidal deformation (stress or strain) is applied to the material through the oscillation of the top plate at a fixed frequency ω. The material response (strain or stress) is then measured.



Let's assume that a sinusoidal strain γ^* is applied and that the stress τ^* is measured, both of them can be expressed using the complex exponential function: $\gamma^* = \gamma_0 e^{i\omega t}$, $\tau^* = \tau_0 e^{i(\omega t+\delta)}$.

With δ the phase shift between the deformation and the response ($\delta = 0$ for a purely elastic material, $\delta = \frac{\pi}{2}$ for a purely viscous one, and $0 < \delta < \frac{\pi}{2}$ for viscoelastic materials).

4a. In the linear regime, recall that the following relation links γ^* and τ^*: $\tau^* = G^*\gamma^*$ where $G^* = G' + iG''$ is the complex modulus. Show that:

$$G' = \frac{\tau_0}{\gamma_0}\cos(\delta) \text{ and } G'' = \frac{\tau_0}{\gamma_0}\sin(\delta)$$

4b. G' and G'' are respectively the storage and loss modulus.

(i) In order to measure the material damping properties, the value of $\tan(\delta)$ can be computed. Show that $\tan(\delta) = \frac{G''}{G'}$.

(ii) What value of $\tan(\delta)$ would you expect for a purely elastic material ? and a purely viscous material ? If you consider now a viscoelastic material, what can you conclude on the meaning of G' and G''?

4c. Express the value of the complex viscosity extracted from oscillatory rheology as a function of G' and G'' and conclude that it can be written as $\eta^* = \eta' + i\eta''$.
(Recall: $\tau = \eta \times \dot{\gamma} = \eta \times \frac{d\gamma}{dt}$).

4d. Show that $|\eta^*| = \frac{|G^*|}{\omega}$

4e. Now assume that you are applying a sinusoidal stress $\tau^* = \tau_0 e^{i\omega t}$ and you measure the response strain $\gamma^* = \gamma_0 e^{i(\omega t+\delta)}$. Compute once again the complex modulus G^* and the complex viscosity η^* and show that we still have $|\eta^*| = \frac{|G^*|}{\omega}$.